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The equilibrium configuration of a slowly rotating mass of liquid in the presence of a poloidal magnetic field

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The equilibrium configuration of a slowly rotating self-gravitating perfectly conducting inviscid liquid, in the presence of a small poloidal magnetic field, is considered for a case where the electric current is a simple function of the distance from the axis of rotation. Owing to the coupling of the magnetic field with the rotation the electric current may reverse direction. This could make the magnetic field zero on certain surfaces and impose restrictions on the parameters of the problem. A perturbation expansion of the nearly spherical surface of the liquid is constructed.

1. Introduction

The problem of the mechanical equilibrium of a perfectly conducting self-gravitating liquid in the presence of a magnetic field, owing to its astrophysical applications, has been considered by several authors. For obvious reasons all investigations have been restricted to the case where the velocity and magnetic field have axial symmetry. Chandrasekhar (1956) expressed the equations governing the general case of axial symmetry in a most convenient form. Special solutions of these equations have since been considered by several authors for various configurations. For example, some cases were investigated by means of the variational method by Woltjer (1959) and Wentzel (1960). These solutions, however, are based on the existence of surface currents. More recently Sozou (1972) showed that if the effects of surface currents are ignored a solution exists where the fluid is a Maclaurin spheroid. The magnetic field enables the interior of the spheroid to have a differential angular velocity.

The more realistic problem where the magnetic field is continuous at the surface is difficult. For this reason analytic solutions of this problem are restricted to the case where the magnetic field is weak and the fluid is rotating slowly and thus has a nearly spherical surface. Ferraro (1954) was the first to consider this problem. He considered the case of a liquid star rotating slowly with constant angular velocity in the presence of a weak poloidal magnetic field, finite everywhere, and showed that the magnetic field increases the small eccentricity of a rotating Maclaurin spheroid. Roberts (1955) attempted to extend Ferraro's work and constructed a series expansion for the liquid surface in the presence of Ferraro's field. As was pointed out by Wentzel (1960), however, the Roberts series, excepting the first-order term, is incorrect.

Chiam & Monaghan (1971) considered the structure of a weak poloidal magnetic field in a star for the case where there is a dipole field at the origin. Their approximate solution depends on eigenvalues and is not unique. Unique solutions can be obtained when the electric current density is an explicit function of the spatial co-ordinates and does not depend on the boundary shape. For a liquid rotating about an axis in the presence of a poloidal magnetic field this happens only when the electric current density is proportional to $A\varpi - K\varpi^3$, where A and K are constants and ϖ is the distance from the axis of rotation. The case $K = 0$ is that considered by Ferraro. The part $K\varpi^3$ of the current is entirely due to the differential rotation of the fluid. The magnetic field associated with the case $A \neq 0$, $K \neq 0$ may be considered as a generalized Ferraro field. In this note we consider the effect of this generalized field on the equilibrium configuration of a nearly spherical, self-gravitating mass of liquid and show that a series expansion may be constructed for the free surface of the liquid.

2. Equations of the problem

We consider a perfectly conducting incompressible fluid of density ρ occupying a finite region of space. Our analysis refers to a steady-state axisymmetric configuration with the axis of symmetry along the z axis of a cylindrical polar co-ordinate system (ϖ, ϕ, z). We assume that the fluid velocity $\mathbf{v}$ has only an azimuthal component and that the magnetic field $\mathbf{B}$ lies entirely in meridian planes containing the axis of symmetry, that is, we assume that

$$\mathbf{v} = \hat{\boldsymbol{\phi}}\varpi f, \tag{1}$$

$$\mathbf{B} = \nabla \times [-(\chi/\varpi)\,\hat{\boldsymbol{\phi}}], \tag{2}$$

where f is a function of position and χ is the magnetic stream function.

It was shown by Ranger (1970) that if $f = f(\chi)$ the equation of motion (momentum equation) is satisfied when χ satisfies the equation

$$D^2\chi + \rho\mu_0\varpi^4 f\,df/d\chi = \mu_0\varpi^2 F(\chi), \tag{3}$$

where μ_0 is the magnetic permeability of the fluid, F is an arbitrary function and

$$D^2 = \frac{\partial^2}{\partial\varpi^2} - \frac{1}{\varpi}\frac{\partial}{\partial\varpi} + \frac{\partial^2}{\partial z^2}.$$

Equation (3) is a special case of a more general solution considered by Chandrasekhar (1956).

On integrating the momentum equation we find that the pressure p and gravitational potential Ω of the fluid are related by

$$p + \rho\Omega = \tfrac{1}{2}\rho\varpi^2 f^2 - \int F(\chi)\,d\chi. \tag{4}$$

Since at the free surface of the fluid p is zero, we must have

$$\rho\Omega_s = \tfrac{1}{2}\rho\varpi_s^2 f^2(\chi_s) - \int^{\chi_s} F(\chi)\,d\chi + \text{constant}, \tag{5}$$

where s denotes a surface value.

The electric current $\mathbf{j}$ has only an azimuthal component and is given by

$$\mu_0 \mathbf{j} = \nabla \times \mathbf{B} = \mu_0 j \hat{\boldsymbol{\phi}} = \frac{1}{\varpi} D^2 \chi \hat{\boldsymbol{\phi}} = \nabla^2 \left(\frac{\chi}{\varpi} \hat{\boldsymbol{\phi}} \right). \tag{6}$$

Thus if we assume that $\mathbf{B}$ is entirely due to the electric currents in the fluid

$$\hat{\boldsymbol{\phi}} \chi = -\frac{\mu_0}{4\pi} \varpi \int \frac{j \hat{\boldsymbol{\phi}}' \, d\tau'}{|\mathbf{r} - \mathbf{r}'|},$$

or

$$\chi = -\frac{\mu_0}{4\pi} \varpi \int \frac{j(\varpi', z') \cos(\phi - \phi') \, d\tau'}{|\mathbf{r} - \mathbf{r}'|}, \tag{7}$$

where the integration is taken over the volume of the fluid, $\mathbf{r}$ denotes the radial distance from the origin and a prime refers to the value of the quantity in $d\tau'$.

Equations (3) and (6) show that the electric current is an explicit function of ϖ and z provided that

$$F(\chi) = A \tag{8}$$

and

$$f\, df/d\chi = K/\rho, \quad \text{or} \quad f^2 = 2K\chi/\rho + \omega_0^2, \tag{9}$$

where A, K and ω_0 are constants. If we now make use of (3) and (6)–(9), after a little algebra, (5) becomes

$$-\rho \Omega_s - (A - \varpi_s^2 K) \chi + \tfrac{1}{2} \rho \varpi_s^2 \omega_0^2 = \text{constant}, \tag{10}$$

or

$$\int \frac{d\tau'}{|\mathbf{r} - \mathbf{r}'|} + \frac{\mu_0 (A \varpi_s - K \varpi_s^3)}{4\pi \rho^2 G} \int \frac{(A\varpi' - K\varpi'^3) \cos(\phi_s - \phi') \, d\tau'}{|\mathbf{r}_s - \mathbf{r}'|} + \frac{\omega_0^2 \varpi_s^2}{2\rho G} = \text{constant}, \tag{11}$$

where G is the gravitational constant. Roberts (1955) considered the special case $K = 0$, $\omega_0 = 0$ and expanded the integrals occurring in (11) in terms of Legendre polynomials. He then restricted his analysis to the case where the magnetic energy is much less than the gravitational energy of the fluid and constructed a series expansion for the nearly spherical surface of the fluid. As was pointed out by Wentzel (1960), Roberts's expansion of the integrals in (11), in terms of Legendre polynomials, is incorrect and consequently his series for the fluid surface is also incorrect except for the first-order term.

3. Solution for a nearly spherical boundary

Equation (11) is a rather complicated equation and cannot be tackled, in general, except numerically. For this reason we shall restrict our analysis to the case where the magnetic and kinetic energies of the system are much less than its potential energy, and the fluid surface is nearly spherical. For this case it is convenient to express Ω, χ and r_s in terms of Legendre polynomials and, in order to avoid integrations, use (10) instead of (11). Equation (10) is solved by means of a regular perturbation as follows.

For a nearly spherical surface of approximate radius a we assume that

$$\frac{A^2 \mu_0}{4\pi a^2 \rho^2 G} \ll 1, \quad \frac{a^2 K^2 \mu_0}{4\pi \rho^2 G} \ll 1, \quad \frac{\omega_0^2}{4\pi \rho G} \ll 1$$

and set

$$A \mu_0^{\frac{1}{2}} = \lambda \lambda_1 \epsilon^{\frac{1}{2}}, \quad aK \mu_0^{\frac{1}{2}} = \lambda \lambda_2 \epsilon^{\frac{1}{2}}, \quad \omega_0^2 = \lambda^2 \lambda_3^2 \epsilon / \rho, \tag{12}$$

where $\lambda = 2a\rho\pi^{\frac{1}{2}}G^{\frac{1}{2}}$, ϵ is a small quantity and λ_1, λ_2 and λ_3 are constants of order unity. Using spherical polar co-ordinates (r, θ, ϕ) we set

$$r_s = R(\mu) = a[1+\epsilon\Sigma a_{1i}P_i(\mu)+\epsilon^2\Sigma a_{2j}P_j(\mu)+\dots] \qquad (i = 2,4,6;\ j = 2,4,\dots,12), \tag{13}$$

$$\Omega = C+\tfrac{2}{3}\pi\rho G[r^2+\epsilon\Sigma b_{1i}r^iP_i+\epsilon^2\Sigma b_{2j}r^jP_j+\dots], \tag{14a}$$

$$\Omega = -\tfrac{4}{3}\pi\rho a^3G[r^{-1}+\epsilon r^{-1}\Sigma c_{1i}r^{-i}P_i+\epsilon^2 r^{-1}(c_{20}+\Sigma c_{2j}r^{-j}P_j)+\dots], \tag{14b}$$

$$\chi = (1-\mu^2)\mu_0[d_{11}r^2+\tfrac{1}{10}Ar^4-\tfrac{1}{35}Kr^6+(d_{13}r^4+\tfrac{1}{135}Kr^6)P_3' + \epsilon(d_{21}r^2+d_{23}r^4P_3'+\dots+d_{29}r^{10}P_9')+\dots], \tag{15a}$$

$$\chi = (1-\mu^2)\mu_0\left[\frac{e_{11}}{r}+\frac{e_{13}}{r^3}P_3'+\epsilon\left(\frac{e_{21}}{r}+\frac{e_{23}}{r^3}P_3'+\dots\frac{e_{29}}{r^9}P_9'\right)+\dots\right], \tag{15b}$$

where C is a constant, P_n is a Legendre polynomial of degree n, $\mu = \cos\theta$, (14a) and (15a) hold for the fluid region and (14b) and (15b) for the region exterior to the fluid. The terms involving A and K in (15) belong to the particular integral of (3) and the other terms belong to the complementary function of (3).

The various constants in (13)–(15) are determined from the fact that at the boundary, given by (13), Ω, χ and their normal derivatives are continuous, and (10) is satisfied. The continuity of the normal derivatives of Ω and χ, on the surface given by (13), coupled with the continuity of Ω and χ, to a certain order in ϵ, implies continuity of $\partial\Omega/\partial r$ and $\partial\chi/\partial r$.

First approximation

The continuity of Ω and $\partial\Omega/\partial r$ to order ϵ requires that

$$\left.\begin{aligned} 6a_{12} &= -5b_{12} = 10a^{-2}c_{12},\\ 2a_{14} &= -3a^2b_{14} = 6a^{-4}c_{14},\\ 6a_{16} &= -13a^4b_{16} = 6a^{-6}c_{16},\end{aligned}\right\} \tag{16}$$

and the continuity of χ and $\partial\chi/\partial r$ to order $\epsilon^{\frac{1}{2}}$ requires that

$$\left.\begin{aligned} d_{11} &= -\tfrac{1}{6}Aa^2+\tfrac{1}{15}Ka^4, \quad d_{13} = -\tfrac{1}{105}Ka^2,\\ e_{11} &= -\tfrac{1}{15}Aa^5+\tfrac{4}{105}Ka^7, \quad e_{13} = -\tfrac{2}{945}Ka^9.\end{aligned}\right\} \tag{17}$$

Equation (9) is a manifestation of the law of isorotation and implies that individual field lines, given by χ = constant, rotate with (individual) constant angular velocity. Thus the fluid surface, where χ is continuous and not constant, has a differential angular velocity.

If we now note that $\varpi_s^2 = R^2(1-\mu^2)$ and, using (12), (16) and (17), substitute (13), (14) and (15) into (10), after a little algebra we obtain, to order ϵ, an equation of the form

$$A_2P_2+A_4P_4+A_6P_6 = \text{constant},$$

where the A's are constants involving λ_1, λ_2, λ_3 and linear combinations of a_{12}, a_{14} and a_{16}. On equating the A's to zero we find that

$$a_{12} = -\tfrac{1}{3}\lambda_1^2+\tfrac{88}{147}\lambda_1\lambda_2-\tfrac{104}{441}\lambda_2^2-\tfrac{5}{2}\lambda_3^2, \quad a_{14} = -\frac{1144\lambda_1\lambda_2-712\lambda_2^2}{13475}, \quad a_{16} = -\frac{104\lambda_2^2}{24255}. \tag{18}$$

If $\lambda_2 = 0$, that is if $f = \omega_0$ and the fluid is rotating with constant angular velocity,

$$a_{12} = -\tfrac{1}{3}\lambda_1^2 - \tfrac{5}{2}\lambda_3^2, \quad a_{14} = a_{16} = 0,$$

and the equation of the free surface is

$$R = a[1 - (\tfrac{1}{3}\lambda_1^2 + \tfrac{5}{2}\lambda_1^2)\, P_2(\mu)],$$

which is that of a planetary ellipsoid. This is the case considered by Ferraro (1954). When $\lambda_2 \neq 0$, a_{16} and, except for the case $143\lambda_1 = 89\lambda_2$, a_{14} are not zero and the free fluid surface is not that of a planetary ellipsoid. Owing to the coupling of the angular velocity with the magnetic field the right-hand side of (9) will not be positive unless constraints are imposed on λ_1, λ_2 and λ_3. In the fluid region the first approximation to χ may be written as

$$\chi = \tfrac{1}{630}\mu_0 r^2(1-\mu^2)\,[42a^4K - 105a^2A + (63A - 36a^2K)r^2 + 10Kr^4 + K(45a^2 - 35r^2)r^2(1-\mu^2)]. \quad (19)$$

$K\chi$ and the right-hand side of (9) will be positive irrespective of the value of ω_0 if for $a > r$ the expression

$$42a^4K - 105a^2A + (63A - 36a^2K)r^2 + 10Kr^4 \quad (20)$$

is positive for positive K and negative for negative K.

Let us consider the case $K > 0$. The minimum of (20) occurs when

$$r^2 = (36a^2K - 63A)/20K.$$

Expression (20) will be positive for $0 < r < a$ (i) if it is positive at $r = a$ and the minimum occurs at $r > a$, that is if

$$16Ka^2 - 42A > 0, \quad 16a^2K - 63A > 0, \quad (21)$$

or if (ii) its minimum value is positive and (20) has no real roots, that is if

$$40K(42a^4K - 105a^2A) - (36a^2K - 63A)^2 > 0. \quad (22)$$

The inequalities (21) are satisfied if $A < 16a^2K/63$. Inequality (22) is satisfied if

$$\frac{8 - 8(55)^{\frac{1}{2}}}{189} < A < \frac{8 + 8(55)^{\frac{1}{2}}}{189}a^2K = 0{\cdot}356a^2K.$$

Therefore, if $K > 0$, (20) will be positive for $0 < r < a$ when $A < 0{\cdot}356a^2K$. Similarly when $0 > K$, (20) will be negative for $0 < r < a$ provided that

$$A > -0{\cdot}356a^2K.$$

Thus $K\chi$ will be positive provided that

$$KA < 0{\cdot}356a^2K^2. \quad (23)$$

Note that if (23) is violated χ is not necessarily of constant sign. A change in sign of χ implies a change in the direction of the magnetic field. Thus, depending on the magnitude of $|A|$ relative to $|K|$, **B** may have a somewhat complex structure.

If, to the first-order approximation, we express $R(\mu)$ as a third-degree polynomial in μ^2 and make use of (23) we can show that, when $\omega_0 = 0$, $R(\mu)$ decreases monotonically from $\mu = 0$ to $\mu = 1$. It can also be shown that when the above inequalities are not satisfied and $K\chi$ becomes negative the magnitude of ω_0, which must be non-zero to make the right-hand side of (9) positive, is such that $R(\mu)$ must decrease monotonically from $\mu = 0$ to $\mu = 1$. Thus the fluid surface, though not spheroidal, is still of a planetary form with its minimum radius at the poles and maximum radius at the equator.

Second approximation

Since $\partial\chi/\partial r$ is continuous to order $\epsilon^{\frac{1}{2}}$, the continuity of χ to order $\epsilon^{\frac{3}{2}}$ requires that

$$d_{2i}a^{2i+1} = e_{2i} \quad (i = 1, 3, \ldots, 9). \tag{24}$$

On using (17) and (24) we find that the continuity of $\partial\chi/\partial r$ to order $\epsilon^{\frac{3}{2}}$ requires that

$$\sum_0^4 (4n+3)\, a^{2n}\, d_{2(2n+1)} P'_{2n+1} + (Aa^2 - \tfrac{4}{5}Ka^4 + \tfrac{2}{15}Ka^4P'_3)\,(a_{12}P_2 + a_{14}P_4 + a_{16}P_6) = 0. \tag{25}$$

Equation (25) may be expressed as an equation of the form

$$A_0 + A_2P_2 + \ldots A_8P_8 = 0$$

or as a fourth-degree equation in μ^2. $d_{21}, d_{23}, \ldots, d_{29}$ can be evaluated by setting $A_0 = A_2 = \ldots = A_8 = 0$ or by equating to zero the coefficients of each power of μ.

The continuity of Ω to order ϵ^2 implies that

$$-3(a_{12}P_2 + a_{14}P_4 + a_{16}P_6)^2 + \Sigma(a^{i-2}b_{2i} + 2a^{-i}c_{2i})\,P_{2i} = \text{constant} \quad (i = 2, 4, \ldots, 12). \tag{26}$$

This may be expressed in the form

$$A_2P_2 + A_4P_4 + \ldots + A_{12}P_{12} = 0$$

and on equating to zero the A's we obtain six equations of the form

$$a^{i-2}b_{2i} + 2a^{-i}c_{2i} = f_{2i} \quad (i = 2, 4, \ldots, 12), \tag{27}$$

where the f's are constants. On doing a similar analysis for the continuity of $\partial\Omega/\partial r$ and the satisfaction of (10), to order ϵ^2, we obtain

$$c_{20} = g_{20}, \tag{28}$$

$$6a_{2i} + ia^{i-2}b_{2i} - 2(i+1)\,a^{-i}c_{2i} = g_{2i} \quad (i = 2, 4, \ldots, 12) \tag{29}$$

and

$$2a_{2i} + a^{i-2}b_{2i} = h_{2i}, \tag{30}$$

where the g's and the h's are constants. c_{20} is readily determined from (28); c_{20} is needed for the next order approximation. a_{2i}, b_{2i} and c_{2i} are easily calculated from (27), (29) and (30). The main difficulty is the tedious computation of f_{2i}, g_{2i} and h_{2i}. For this reason we restrict our computations for this and the third approximation to the special case where $K = 0 = \lambda_2$. Then

$$a_{14} = a_{16} = 0 = a_{2j} = b_{2j} = c_{2j} \quad (j = 6, 8, 10, 12),$$

and

$$d_{21} = a^{-3}e_{21} = \tfrac{1}{15}Aa^2a_{12}, \quad d_{23} = a^{-7}e_{23} = -\tfrac{1}{35}Aa_{12},$$
$$d_{2i} = e_{2i} = 0 \quad (i > 3).$$

After a little algebra we find that

$$a_{22} = (\tfrac{18}{49}\lambda_1^2 + \tfrac{20}{7}\lambda_3^2)\,a_{12}, \quad a_{24} = -(\tfrac{207}{1225}\lambda_1^2 + \tfrac{27}{14}\lambda_3^2)\,a_{12},$$
$$b_{22} = -(\tfrac{122}{245}\lambda_1^2 + \tfrac{27}{7}\lambda_3^2)\,a_{12}, \quad a^2 b_{24} = -\tfrac{72}{1225}\lambda_1^2 a_{12},$$
$$c_{20} = \tfrac{3}{5}a_{12}^2, \quad a^{-2}c_{22} = (\tfrac{26}{245}\lambda_1^2 + \tfrac{6}{7}\lambda_3^2)\,a_{12}, \quad a^{-4}c_{24} = -(\tfrac{279}{1225}\lambda_1^2 + \tfrac{27}{14}\lambda_3^2)\,a_{12}.$$

Third approximation

The complexity of the algebraic operations involved increases very rapidly with the order of the approximation and the process of reducing the expressions obtained to Legendre polynomials is lengthy and tedious. For this reason this is the last approximation we shall carry out. On carrying out an analysis similar to that for the second approximation we find that

$$d_{31} = (\tfrac{143}{4410}\lambda_1^2 + \tfrac{1}{4}\lambda_3^2)\,Aa^2 a_{12}, \quad d_{33} = -(\tfrac{113}{8575}\lambda_1^2 + \tfrac{11}{98}\lambda_3^2)\,Aa_{12},$$
$$d_{35} = -\frac{12\lambda_1^2 A a^{-2} a_{12}}{13475}, \quad e_{31} = (\tfrac{124}{2205}\lambda_1^2 + \tfrac{3}{7}\lambda_3^2)\,Aa^5 a_{12},$$
$$e_{33} = a^7 d_{33}, \quad e_{35} = (\tfrac{349}{40425}\lambda_1^2 + \tfrac{1}{14}\lambda_3^2)\,Aa^9 a_{12}.$$

From the continuity of Ω and $\partial\Omega/\partial r$ and the satisfaction of (10) to order ϵ^3 we obtain a set of equations in a_{3i}, b_{3i} and c_{3i} having a structure identical to that of (27)–(30). The unknowns a_{3i}, b_{3i} and c_{3i} and constants f_{3i}, g_{3i} and h_{3i} occupy the positions corresponding to a_{2i}, b_{2i}, c_{2i}, f_{2i}, g_{2i} and h_{2i}, respectively. [It is easy to show that for an nth-order approximation a_{ni}, b_{ni}, c_{ni} and the constants f_{ni}, g_{ni} and h_{ni} satisfy a set of equations having a structure identical to that of (27)–(30).] After some algebra we find that

$$\begin{bmatrix} a_{32} \\ a_{34} \\ a_{36} \end{bmatrix} = \begin{bmatrix} 0{\cdot}2511 & 3{\cdot}559 & 12{\cdot}98 \\ -0{\cdot}0961 & -2{\cdot}017 & -10{\cdot}77 \\ 0{\cdot}0095 & 0{\cdot}473 & 3{\cdot}65 \end{bmatrix} \Lambda,$$

$$\begin{bmatrix} b_{32} \\ a^2 b_{34} \\ a^4 b_{36} \end{bmatrix} = \begin{bmatrix} -0{\cdot}3431 & -4{\cdot}833 & -17{\cdot}45 \\ -0{\cdot}0742 & -0{\cdot}650 & 0 \\ -0{\cdot}0052 & 0 & 0 \end{bmatrix} \Lambda,$$

$$\begin{bmatrix} c_{30} \\ a^{-2}c_{32} \\ a^{-4}c_{34} \\ a^{-6}c_{36} \end{bmatrix} = \begin{bmatrix} -0{\cdot}1406 & -2{\cdot}150 & -8{\cdot}14 \\ 0{\cdot}1625 & 2{\cdot}440 & 9{\cdot}41 \\ -0{\cdot}0768 & -1{\cdot}264 & -5{\cdot}51 \\ 0{\cdot}1054 & 1{\cdot}842 & 8{\cdot}04 \end{bmatrix} \Lambda,$$

where Λ is the column vector $(a_{12}\lambda_1^4,\ a_{12}\lambda_1^2\lambda_3^2,\ a_{12}\lambda_3^4)$.

Roberts, who dealt with the case $\lambda_1 = 1$, $\lambda_2 = \lambda_3 = 0$, expressed R in the form $R_0(1 + a_1\mu^2 + a_2\mu^4 + a_3\mu^6 + \ldots)$, where R_0 is a constant and the a's are power series in ϵ. From our data it is easy to show that the *correct* expression for the Roberts series, to order ϵ^3, is

$$R = R_0[1 + (-0{\cdot}5\epsilon - 0{\cdot}312\epsilon^2 + 0{\cdot}0668\epsilon^3)\mu^2 + (0{\cdot}246\epsilon^2 + 0{\cdot}133\epsilon^3)\mu^4 - 0{\cdot}0458\epsilon^3\mu^6].$$

We note that only the coefficients of $\epsilon^i\mu^{2i}$ in this series are the same as the corresponding coefficients in the series constructed by Roberts.

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